2°/ $f(x)=\dfrac{x^3}{\sqrt{1+x^4}}$ on pose $u(x)=1+x^4$, $u'(x)=4x^3$ d'où $\forall x\in\mathbf{R}$,

$f(x)=\dfrac{1}{4}\dfrac{u'(x)}{\sqrt{u(x)}}\Rightarrow \mathrm{F}(x)=\dfrac{1}{2}\sqrt{u(x)}+k=\dfrac{\sqrt{1+x^4}}{2}+k$.

or $\mathrm{F}(0)=0\Rightarrow k+\dfrac{1}{2}=0\Rightarrow k=\dfrac{-1}{2}$.

Conclusion : $\forall x\in\mathbf{R}$, $\mathrm{F}(x)=\dfrac{\sqrt{1+x^4}}{2}-\dfrac{1}{2}$.

Solution 13 :

1°/ $e^{ix}=\cos x+i\sin x$; $e^{-ix}=\cos x-i\sin x$.

On a : $\cos 3x=\dfrac{e^{i3x}+e^{-i3x}}{2}$ et $\sin x=\dfrac{e^{ix}-e^{-ix}}{2i}$

d'où $f(x)=(\dfrac{e^{i3x}+e^{-i3x}}{2})(\dfrac{e^{ix}-e^{-ix}}{2i})^3$

$$=\frac{(e^{i3x}+e^{-i3x})(e^{i3x}-3e^{ix}+3e^{-ix}-e^{-i3x})}{-16i}$$

$$=\frac{e^{i6x}-3e^{i4x}+3e^{i2x}-1+1-3e^{-2ix}+3e^{-i4x}-e^{-i6x}}{-16i}$$

$$=\frac{(e^{i6x}-e^{-i6x})-3(e^{i4x}-e^{-i4x})+3(e^{i2x}-e^{-i2x})}{-16i}$$

$$=\frac{2i\sin 6x-3(2i\sin 4x)+3(2i\sin 2x)}{-16i}$$

d'où : $f(x)=\dfrac{-1}{8}\sin 6x+\dfrac{3}{8}\sin 4x-\dfrac{3}{8}\sin 2x$.

2°/ $\mathrm{F}(x)=\dfrac{-1}{8}(\dfrac{-1}{6}\cos 6x)+\dfrac{3}{8}(\dfrac{-1}{4}\cos 4x)-\dfrac{3}{8}(\dfrac{-1}{2}\cos 2x)+k$

$\mathrm{F}(x)=\dfrac{1}{48}\cos 6x-\dfrac{3}{32}\cos 4x+\dfrac{3}{16}\cos 2x+k$.

Or $\mathrm{F}(0)=0\Rightarrow \dfrac{1}{48}-\dfrac{3}{32}+\dfrac{3}{16}+k=0\Rightarrow k=\dfrac{-11}{96}$.

Conclusion : $\forall x\in\mathbf{R}$, $\mathrm{F}(x)=\dfrac{1}{48}\cos 6x-\dfrac{3}{32}\cos 4x+\dfrac{3}{16}\cos 2x-\dfrac{11}{96}$.